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World-Sheet Supersymmetry Without Contact Terms⋆

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Green and Seiberg showed that, in simple treatments of fermionic string theory, it is necessary to introduce contact interactions when vertex operators collide. Otherwise, certain superconformal Ward identities would be violated. In this note, we show how these contact terms arise naturally when proper account is taken of the superconformal geometry involved when punctures collide. More precisely, we show that there is no contact term at all! Rather, corrections arise to the "naïve" formula when the boundary of moduli space is described correctly.

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The existence of contact terms in string theory seems, after a moment's thought, to be somewhat paradoxical. In the *stable compactification* of moduli space [1,2], punctures (*i.e.* the locations at which vertex operators are inserted) are, *by definition* not allowed to collide. Rather, as two punctures approach each other, we can by a conformal transformation take the punctures to remain separated while a piece of the surface surrounding them pinches off (Fig. 1). If the punctures never collide, there would seem to be little role for a δ-function contact term. A more sophisticated person might replace the notion of a contact term when punctures coincide with that of a contribution to the string measure which is "*δ-function-concentrated on the boundary of moduli space corresponding to the pinched surface.*" But this formulation seems rather strained and hard to derive from conformal field theory. After all, conformal field theory amplitudes are supposed to be real-analytic functions of the moduli, which a δ-function certainly is not.

Fig. 1:Colliding punctures.

Fortunately, it now appears that "contact interactions" have a perfectly natural home in string theory. What appear to be contact interactions arise as a consequence of certain global features of the moduli space of punctured Riemann surfaces. The important point is that in order to define the operators which go into a conformal field theory correlation function, we need, implicitly, a normal-ordering prescription or, what amounts to the same thing, a choice of a local coordinate at the insertion point. Actually, we need a family of such local coordinates which vary as we vary the moduli of the surface. Our standard beliefs about the properties of string amplitudes obtain when the coordinate families vary holomorphically with the moduli. Unfortunately, there are obstructions to finding such a holomorphic coordinate family globally on moduli space. We are forced to allow our normal-ordering prescription to vary non-holomorphically with the moduli.

One consequence of this phenomenon is that the one-point function of the zero momentum dilaton vertex operator (which naïvely decouples) is not zero, but rather is proportional to the Euler characteristic of the Riemann surface [3,4,5]. In the presence of other vertex operator insertions, one needs a more refined treatment of the behavior of coordinate families as punctures collide. This leads to the dilaton insertion being proportional to the Euler characteristic of the *punctured* Riemann surface [6]. In other words, the dilaton insertion counts the powers of the string coupling in the correlation function, or, if you wish, there is a low-energy theorem $D \sim \lambda \frac{\partial}{\partial \lambda}$. For the bosonic string, this is true only for the first variation with respect to λ (one dilaton insertion), but for the heterotic string, it holds to all orders (multiple dilaton insertions) [7]. In similar fashion, one can recover the "dilaton" [8] and "puncture" [9] equations of topological gravity, postulated in [10,11]. The key point to be emphasized here is that none of these results involve mysterious δ-functions at the boundary of moduli space. The string measure is perfectly smooth and, aside from physical on-shell poles, vanishes as one approaches the boundary of moduli space.

Another sort of contact term which one encounters in fermionic string theory is that proposed by Green and Seiberg [12]. The "integrated" form of a NS vertex operator (*i.e.* when one thinks of the vertex operator as a 2-form to be integrated over the Riemann surface) is naturally in the (0)-picture. However, if one naïvely takes the OPE of two such vertex operators as the approach each other, the result violates the superconformal Ward identities. The "solution" is to add a δ-function contact term which restores the Ward identities. Green and Seiberg showed that (at least for free field theories) this contact term could be explained by including the auxiliary fields, F, necessary to close the world-sheet supersymmetry algebra off-shell. The equation of motion for the auxiliary field is simply $F = 0$, and the auxiliary field doesn't propagate – its two-point function is just a δ-function. But, noted Green and Seiberg, if we include the contribution of the auxiliary field to the (0)-picture vertex operator, the extra terms which arise from contracting the auxiliary fields when the vertex operators collide precisely reproduce the needed contact terms.

This mechanism, though pretty, has some drawbacks. First, it only applies to field theories in which one knows an off-shell auxiliary field structure. This leaves out a large class of interesting theories which are only superconformal on-shell. One would like to be able to treat more general superconformal theories. Second, by explicitly invoking off-shell features of the theory, one is *potentially* opening a Pandora's box of violations of superconformal invariance. For free theories, it appears that one can easily snap shut the lid after extracting the desired contact terms. For more general theories, even if their auxiliary field structure is known, one may not be so lucky.

In the operator formalism, a vertex operator is naturally thought of as a 0-form inserted at a puncture; it always appears in the (-1)-picture. The picture-changing operator which moves it into the (0)-picture appears when one integrates over the supermodulus associated to the location of the puncture. When two vertex operators collide, we should not be too cavalier in our treatment of picture-changing. We shall see in this paper that in a more careful treatment of the collision of two punctures, extra terms arise in addition to the usual "picture-changing" terms. These are precisely what is necessary to restore the superconformal Ward identities. Thus in a careful treatment of the degenerating surface (Fig. 1), no extra "contact terms" are necessary; all of the necessary corrections are automatically taken into account. All of this can be carried through for an arbitrary superconformal field theory. Indeed, we won't specify precisely what superconformal theory we are working with.

1. Preliminaries

1.1. The Measure

As discussed above, the definition of the normal-ordered vertex operators, and hence of the string measure, depends on a choice of local coordinate at the insertion points. The operator formalism provides a nice framework for keeping track of this dependence and constructing explicit expressions for the superstring measure. The extra data of a local coordinate at each puncture fits together into an infinite dimensional fiber bundle $\widehat{\mathcal{P}}_{g,n} \to \widehat{\mathcal{M}}_{g,n}$. Choosing a family of local coordinates at the punctures amounts to choosing a section $\sigma : \widehat{\mathcal{M}}_{g,n} \to \widehat{\mathcal{P}}_{g,n}$. The measure on $\widehat{\mathcal{M}}_{g,n}$ is defined by

$$\begin{aligned}\Omega(v_1, v_2, \dots, \bar{v}_{3g-3+n}, \nu_1, \nu_2, \dots, \nu_{2g-2+n}) = \\ \langle\Sigma, (z_1, \theta_1), \dots | B[\sigma_* v_1] \dots B[\sigma_* \bar{v}_{3g-3+n}] \\ \times \delta(B[\sigma_* \nu_1]) \dots \delta(B[\sigma_* \nu_{2g-2+n}]) |\psi_1\rangle_{P_1} \otimes \cdots \otimes |\psi_n\rangle_{P_n}.\end{aligned} \tag{1.1}$$

where $\sigma_* v_i$ is the push-forward of a tangent vector to $\widehat{\mathcal{M}}_{g,n}$ to a tangent vector to $\widehat{\mathcal{P}}_{g,n}$. This construction, and the above notation, are described in great detail in [13,5] and [7].

The tangent space to $\widehat{\mathcal{P}}_{g,n}$ has an elegant presentation in terms of Schiffer variations. It has the structure of a direct sum of n copies of the Neveu-Schwarz algebra (one for each puncture) modulo a certain "Borel" subalgebra. The generators of this algebra are given in terms of the local coordinates at the puncture by

$$
\begin{aligned}
l_n &\leftrightarrow -z^{n+1}\frac{\partial}{\partial z} - \tfrac{1}{2}(n+1)z^n\theta\frac{\partial}{\partial\theta} \\
g_k &\leftrightarrow \tfrac{1}{2}z^{k+\frac{1}{2}}\big(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z}\big).
\end{aligned}
\tag{1.2}
$$

An infinitesimal change in the coordinate used at a puncture is given by the action of the "vertical" tangent vectors $l_n,\ g_{n+\frac{1}{2}},\ n \geq 0$. These induce a change of the state associated to the punctured surface given by

$$
\langle\Sigma, z - \epsilon z^{n+1} - \tfrac{1}{2}\hat{\epsilon}\theta z^{k+\frac{1}{2}}| = \langle\Sigma, z|\big(1 + \epsilon L_n + \hat{\epsilon}G_k\big) + \cdots . \tag{1.3}
$$

Again, details may be found in [13,5,7].

1.2. The Coordinate Family

The geometry appropriate for computing contact terms in the operator formalism is now well-established. The heterotic case was treated in detail in [7] and we will borrow the results of some of the calculations presented there. Here we will be content to give a brief account of the geometry and the reader should consult [7] for a more complete account.

When the points P and Q are widely separated, it is appropriate to use coordinates at one point that are independent of the location of the other. A convenient choice of coordinate is the so-called superconformal normal-ordered (SCNO) coordinate [5], and letting $(\widetilde{r}, \widetilde{\rho})$ be the location of P, we take as coordinates

$$
\zeta_P(\cdot) = z(\cdot) - \widetilde{r} - \widetilde{\rho}\theta(\cdot), \qquad \check{\zeta}_P(\cdot) = \theta(\cdot) - \widetilde{\rho} \quad . \tag{1.4}
$$

The point P should be thought of as the place where $\zeta_P(\cdot)$ and $\check{\zeta}_P(\cdot)$ vanish. $\zeta_P(\cdot)$ and $\check{\zeta}_P(\cdot)$ are coordinates that are superconformally related to $z(\cdot)$ and $\theta(\cdot)$. For a SCNO coordinate, $\check{\zeta}_P(\cdot)$ is just $D\zeta_P(\cdot)$, where D is the super-derivative $\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$. Coordinates at Q, located at, say, (r, ρ), can be given by similar expressions, with $\widetilde{r}$ and $\widetilde{\rho}$ replaced by r and ρ.

In conformal field theories, the collision of two punctures is replaced by the conformally equivalent picture of the two punctures located on a sphere pinching off from the rest of the surface. In this region SCNO coordinates are not appropriate. Instead, the coordinates of the point P will depend on the location of Q, and they are given by a plumbing fixture construction [14]. This construction is obtained by beginning with a three-punctured super-sphere. We can use $Osp(2,1)$ symmetry to fix the three bosonic coordinates of the punctures at 0, 1, and ∞. Only two of the three fermionic coordinates can be fixed though; the remaining one is the one odd modulus of the three-punctured super-sphere, which we denote τ. Thus we place our three punctures at (0,0), $(1,\tau)$ and $(\infty,0)$. The point at ∞ is sewn onto the rest of the surface at the old location of Q. If w and ξ are the coordinates on the super-sphere, we can write the most general coordinates that vanish at P and Q as

$$w - \widetilde{a}_1 \tau w \xi + \widetilde{a}_2 w^2 - \widetilde{a}_3 \tau w^2 \xi + \cdots \tag{1.5}$$

at (0,0) and

$$(w-1+\tau\xi) - a_1\tau(w-1+\tau\xi)(\xi-\tau) + a_2(w-1+\tau\xi)^2 - a_3\tau(w-1+\tau\xi)^2(\xi-\tau) + \cdots \tag{1.6}$$

at $(1,\tau)$. We have just given the even coordinates here. The odd coordinates are found by demanding that the coordinate transformation from (w,ξ) be superconformal [7]. The minus signs are for later convenience. At ∞ we can simply use $-1/w$ and ξ/w. These coordinates are sewn to the coordinates $\zeta_Q(\cdot)$ and $\check{\zeta}_Q(\cdot)$ by the identifications

$$w = \frac{\zeta_Q(\cdot)}{t^2} \quad , \qquad \xi = -\frac{\check{\zeta}_Q(\cdot)}{t} \quad . \tag{1.7}$$

Thus, we have as coordinates for P and Q, when they are close together,

$$\begin{aligned} \phi_Q(\cdot) =& \frac{\zeta_Q(\cdot)}{t^2} + \widetilde{a}_1 \tau \frac{\zeta_Q(\cdot)\check{\zeta}_Q(\cdot)}{t^3} + \widetilde{a}_2 \frac{\zeta_Q(\cdot)^2}{t^4} + \cdots \\ \phi_P(\cdot) =& \Sigma(\cdot) + a_1 \tau \Sigma(\cdot)\check{\Sigma}(\cdot) + a_2 \Sigma(\cdot)^2 + \cdots , \end{aligned} \tag{1.8}$$

with

$$\Sigma(\cdot) = \Big(\frac{\zeta_Q(\cdot)}{t^2} - 1 - \frac{\tau\check{\zeta}_Q(\cdot)}{t}\Big), \qquad \check{\Sigma}(\cdot) = \Big(\frac{\check{\zeta}_Q(\cdot)}{t} + \tau\Big) \quad . \tag{1.9}$$

Notice that the coordinates for P are now given in terms of r, ρ, t, and τ instead of $\widetilde{r}$ and $\widetilde{\rho}$. These are the coordinates for supermoduli space appropriate to describe the neighborhood of this boundary of supermoduli space. On the overlap between the two patches of supermoduli space, the change of coordinates from r, ρ, t, τ to $r, \rho, \widetilde{r}, \widetilde{\rho}$ is found by demanding that $\phi_P(P)$ (and its superconformal partner $\check{\phi}_P(P)$) vanish at the same point where the coordinate family in the other patch are centered. This happens when $\zeta_Q(P) = t^2$ and $\check{\zeta}_Q(P) = -t\tau$. We find

$$\widetilde{r} = r + t^2 + t\rho\tau, \qquad \widetilde{\rho} = \rho - t\tau \quad . \tag{1.10}$$

The non-split nature of this transformation is ultimately responsible for the corrections to the "naïve" calculation that restores the Ward identities#. This will be seen explicitly below.

Finally, to get a coordinate family appropriate for both when the punctures are widely separated and when they are close together, one should interpolate smoothly between the two behaviors outlined above. We do this by introducing a smooth function $f(|t|)$ that goes from 0 to 1 as $|t|$ goes from 0 to ∞, and using a linear interpolation. Thus, the coordinates that we will use are

$$\begin{aligned} \sigma_P(\cdot) =& \frac{f(|t|)}{t^2}\zeta_P(\cdot) + (1 - f(|t|))\phi_P(\cdot) \\ \sigma_Q(\cdot) =& \frac{f(|t|)}{t^2}\zeta_Q(\cdot) + (1 - f(|t|))\phi_Q(\cdot)\,. \end{aligned} \tag{1.11}$$

The precise shape of the function f will drop out of the calculation. Any coordinate family which interpolates between the required behavior at large $|t|$ and at small $|t|$ will yield the same value for the measure. (In the case of dilaton correlation functions, the measure itself does depend on the interpolating function f; only its integral is independent of f.)

2. The calculation

The state $\langle\Sigma, \sigma_P, \sigma_Q|$ at $r = \overline{r} = 0$ can be expanded as

$$\begin{aligned} \langle\Sigma, \sigma_P, \sigma_Q| =& \langle\Sigma, \sigma_{P_0}, \sigma_{Q_0}|\big(1 + \rho\tau t L_{-1}^{(P)} + \rho\tau\big(\frac{2p_2 - p_1}{t}\big)L_0^{(P)} + 2\big(\rho - \tau t\big)G_{-\frac{1}{2}}^{(P)} + \cdots\big) \\ & \times\big(1 - \frac{\rho\tau\widetilde{p}_1}{t}L_0^{(Q)} + 2\rho G_{-\frac{1}{2}}^{(Q)} + \cdots\big)t^{2(L_0^{(P)} + L_0^{(Q)})}\overline{t}^{2(\bar{L}_0^{(P)} + \bar{L}_0^{(Q)})} \end{aligned} \tag{2.1}$$

where $p_1 = a_1(1-f)$, $\widetilde{p}_1 = \widetilde{a}_1(1-f)$, and $p_2 = a_2(1-f)$. σ_{P_0} and σ_{Q_0} are the equal to σ_P and σ_Q with ρ and τ set to zero, and rescaled by a factor of t^2. The ghost insertions that result from computing the pushforwards of the tangent vectors associated to the moduli corresponding to the locations of the points P and Q by the coordinate family are given

The role of non-split coordinate transformations on supermoduli space in this regard was also noted by H. Verlinde [15].

in [7]. The measure is formed by multiplying these six contributions together (including delta functions for the insertions for ρ and τ). The insertions are (evaluated at $r=\overline{r}=0$)

$$
\begin{aligned}
B[\sigma_*(\frac{\partial}{\partial r})] =& -\left(\frac{1}{t^2}\right)b^{(P)}_{-1} + \left(\frac{2p_1\tau}{t^2}\right)\beta^{(P)}_{-\frac{1}{2}} - \left(\frac{1}{t^2}\right)b^{(Q)}_{-1} + \left(\frac{2\widetilde{p}_1\tau}{t^2}\right)\beta^{(Q)}_{-\frac{1}{2}} + \cdots \\
B[\sigma_*(\frac{\partial}{\partial \overline{r}})] =& -\left(\frac{1}{\overline{t}^2}\right)\overline{b}^{(P)}_{-1} - \left(\frac{1}{\overline{t}^2}\right)\overline{b}^{(Q)}_{-1} + \cdots \\
B[\sigma_*(\frac{\partial}{\partial \rho})] =& \left(-\frac{\rho}{t^2} + \frac{2\tau}{t}\right)b^{(P)}_{-1} + \left(\frac{2}{t} + \frac{2p_1\rho\tau}{t^2}\right)\beta^{(P)}_{-\frac{1}{2}} - \\
& \left(\frac{\rho}{t^2}\right)b^{(Q)}_{-1} + \left(\frac{2}{t} + \frac{2\widetilde{p}_1\rho\tau}{t^2}\right)\beta^{(Q)}_{-\frac{1}{2}} + \cdots \\
B[\sigma_*(\frac{\partial}{\partial t})] =& -\left(\frac{2}{t}\right)b^{(P)}_{-1} + \left(\frac{2\tau}{t} + \frac{4p_1\tau}{t}\right)\beta^{(P)}_{-\frac{1}{2}} + \cdots \\
B[\sigma_*(\frac{\partial}{\partial \overline{t}})] =& -\left(\frac{2}{\overline{t}}\right)\overline{b}^{(P)}_{-1} + \cdots \\
B[\sigma_*(\frac{\partial}{\partial \tau})] =& -\left(\tau\right)b^{(P)}_{-1} - \left(2\right)\beta^{(P)}_{-\frac{1}{2}} + \cdots
\end{aligned}
\tag{2.2}
$$

where the dots represent higher terms ($b_n, n \geq 0$ and $\beta_n, n \geq \frac{1}{2}$) that annihilate strong physical states (SPSs).

To insert a SPS at P and at Q, there are two possible contributions to the measure from the above insertions:

$$
b^{(P)}_{-1}\overline{b}^{(P)}_{-1}\delta[\beta^{(P)}_{-\frac{1}{2}}]b^{(Q)}_{-1}\overline{b}^{(Q)}_{-1}\delta[\beta^{(Q)}_{-\frac{1}{2}}]\,, \qquad b^{(P)}_{-1}\overline{b}^{(P)}_{-1}\delta[\beta^{(P)}_{-\frac{1}{2}}]b^{(Q)}_{-1}\overline{b}^{(Q)}_{-1}\beta^{(Q)}_{-\frac{1}{2}}\delta'[\beta^{(Q)}_{-\frac{1}{2}}]\,. \tag{2.3}
$$

Using the formal identification of $\beta\delta'(\beta)$ and $-\delta(\beta)$, we easily find the contribution from the insertions in (2.2) to be

$$
\frac{t}{|t|^6}b^{(P)}_{-1}\overline{b}^{(P)}_{-1}\delta[\beta^{(P)}_{-\frac{1}{2}}]b^{(Q)}_{-1}\overline{b}^{(Q)}_{-1}\delta[\beta^{(Q)}_{-\frac{1}{2}}], \tag{2.4}
$$

where there has been some cancellation between the two different contributions in (2.3).

Thus we have

$$
\begin{aligned}
&\int \left[\mathrm{d}r\wedge \mathrm{d}\overline{r}\wedge \mathrm{d}t\wedge \mathrm{d}\overline{t}|\mathrm{d}\rho \mathrm{d}\tau\right]\ \langle \Sigma, \sigma_P, \sigma_Q|\cdots B[\sigma_*(\frac{\partial}{\partial r})]B[\sigma_*(\frac{\partial}{\partial \overline{r}})]\delta\left[B[\sigma_*(\frac{\partial}{\partial \rho})]\right] \\
&\qquad \times B[\sigma_*(\frac{\partial}{\partial t})]B[\sigma_*(\frac{\partial}{\partial \overline{t}})]\delta\left[B[\sigma_*(\frac{\partial}{\partial \tau})]\right]|\psi\rangle^P\otimes|\psi\rangle^Q = \\
&\int \left[\mathrm{d}r\wedge \mathrm{d}\overline{r}\wedge \mathrm{d}t\wedge \mathrm{d}\overline{t}|\mathrm{d}\rho \mathrm{d}\tau\right]\ \langle \Sigma, \sigma_{P_0}, \sigma_{Q_0}| \\
&\qquad \times \left(1+\rho\tau t L^{(P)}_{-1} + \rho\tau\left(\frac{2p_2-p_1}{t}\right)L^{(P)}_0 + 2\left(\rho-\tau t\right)G^{(P)}_{-\frac{1}{2}}\right) \\
&\qquad \times \left(1-\frac{\rho\tau\widetilde{p}_1}{t}L^{(Q)}_0 + 2\rho G^{(Q)}_{-\frac{1}{2}}\right)t^{2(L^{(P)}_0+L^{(Q)}_0)}\overline{t}^{2(\bar{L}^{(P)}_0+\bar{L}^{(Q)}_0)} \\
&\qquad \times \left(\frac{t}{|t|^6}\right)b^{(P)}_{-1}\overline{b}^{(P)}_{-1}\delta[\beta^{(P)}_{-\frac{1}{2}}]b^{(Q)}_{-1}\overline{b}^{(Q)}_{-1}\delta[\beta^{(Q)}_{-\frac{1}{2}}]|\psi\rangle^P\otimes|\psi\rangle^Q.
\end{aligned}
\tag{2.5}
$$

The state $b_{-1}\bar{b}_{-1}\delta[\beta_{-\frac{1}{2}}]|\psi\rangle$ has $L_0=\frac{1}{2}$ and $\bar{L}_0=1$ and thus, picking off the term proportional $\rho\tau$, (2.5) becomes

$$\begin{aligned}\int & [\mathrm{d}r\wedge\mathrm{d}\bar{r}\wedge\mathrm{d}t\wedge\mathrm{d}\bar{t}|\mathrm{d}\rho\mathrm{d}\tau] \\ \langle\Sigma,\sigma_{P_0},\sigma_{Q_0}|\rho\tau & \left(tL_{-1}^{(P)}+\frac{(2a_2-a_1-\widetilde{a}_1)(1-f)}{2t}+4tG_{-\frac{1}{2}}^{(Q)}G_{-\frac{1}{2}}^{(P)}\right) \\ & \times\left(\frac{|t|^2}{t}\right)b_{-1}^{(P)}\bar{b}_{-1}^{(P)}\delta[\beta_{-\frac{1}{2}}^{(P)}]b_{-1}^{(Q)}\bar{b}_{-1}^{(Q)}\delta[\beta_{-\frac{1}{2}}^{(Q)}]|\psi\rangle^P\otimes|\psi\rangle^Q.\end{aligned} \tag{2.6}$$

At first sight, the term proportional to $(1-f)$ looks like bad news, since it depends on the shape of the function f, and hence on the details of the coordinate family chosen. Actually, the coefficients a_i and $\widetilde{a}_i$ are not to be taken to be arbitrary. As discussed in [7], a unique choice for the coefficients is singled out by demanding that dilaton insertions behave correctly. There one found that $a_1=a_2=\widetilde{a}_1=-\widetilde{a}_2=-1/4$. (Actually, the $\widetilde{a}_i$ were not determined in [7], but a calculation exactly analogous to the ones presented there fixes these coefficients as well.) With these values for the coefficients, the offending term vanishes, and all dependence on the shape of the coordinate family chosen drops out. *Any* coordinate family which interpolates to the asymptotic behavior (1.5),(1.6) gives the same result for the measure

$$\begin{aligned}\int[\mathrm{d}r\wedge\mathrm{d}\bar{r}\wedge\mathrm{d}t\wedge\mathrm{d}\bar{t}|\mathrm{d}\rho\mathrm{d}\tau]\ \langle\Sigma,\sigma_{P_0},\sigma_{Q_0}|\rho\tau & \left(L_{-1}^{(P)}+4G_{-\frac{1}{2}}^{(Q)}G_{-\frac{1}{2}}^{(P)}\right) \\ & \times|t|^2 b_{-1}^{(P)}\bar{b}_{-1}^{(P)}\delta[\beta_{-\frac{1}{2}}^{(P)}]b_{-1}^{(Q)}\bar{b}_{-1}^{(Q)}\delta[\beta_{-\frac{1}{2}}^{(Q)}]|\psi\rangle^P\otimes|\psi\rangle^Q.\end{aligned} \tag{2.7}$$

Our demand that the coordinate family exhibit a certain asymptotic behavior as we approach the boundary of moduli space has both a physical motivation (it gives the correct behavior for dilaton correlation functions) and a mathematical motivation. When we think about the plumbing construction as gluing a certain *fixture* – a 3-punctured sphere *with* coordinates – onto the rest of the surface, it makes sense to demand that the resulting sewn surface be the same, regardless of which puncture we chose to attach to the rest of the surface. In other words, we demand invariance under (a subgroup of) mœbius transformations of the 3-punctures sphere. As in the bosonic string [6,16], this leads to a definite choice for the first few coefficients in (1.5),(1.6). Still, one might be puzzled that the measure formed with SPSs should be sensitive at all to the asymptotic behavior of the coordinate family. We will defer a discussion of this point to section 3.

Of the two terms in (2.7), the $G_{-\frac{1}{2}}^{(Q)}G_{-\frac{1}{2}}^{(P)}$ is precisely what is expected from a naïve analysis of the collision of two vertex operators. Indeed, the $G_{-\frac{1}{2}}^{(P)}$ combines with the $\delta[\beta_{-\frac{1}{2}}^{(P)}]$

to form a picture-changing operator, which converts the state at P from the (-1) to the (0) picture (and similarly for the state at Q). The $L_{-1}^{(P)}$ term is new and, as we shall see, plays the role of the "contact term" postulated by Green and Seiberg.

First, let us see that it makes sense by examining the transformation of the measure under the coordinate transformation in (1.10). There are three contributions to the change. The first is the how the integration measure $[\mathrm{d}\widetilde{r}\wedge \mathrm{d}\overline{\widetilde{r}}|\mathrm{d}\widetilde{\rho}]$ transforms to $[\mathrm{d}t\wedge \mathrm{d}\bar{t}|\mathrm{d}\tau]$. The change of variables requires the introduction of a Jacobian (Berezinian),

$$[\mathrm{d}\widetilde{r}\wedge \mathrm{d}\overline{\widetilde{r}}|\mathrm{d}\widetilde{\rho}] = \mathrm{sdet}\begin{pmatrix} \frac{\partial\widetilde{r}}{\partial t} & \frac{\partial\widetilde{\rho}}{\partial t} \\ \frac{\partial\widetilde{r}}{\partial\tau} & \frac{\partial\widetilde{\rho}}{\partial\tau}\end{pmatrix} \det\left(\frac{\partial\overline{\widetilde{r}}}{\partial\bar{t}}\right)[\mathrm{d}t\wedge \mathrm{d}\bar{t}|\mathrm{d}\tau]. \tag{2.8}$$

Using (1.10), this results in

$$[\mathrm{d}\widetilde{r}\wedge \mathrm{d}\overline{\widetilde{r}}|\mathrm{d}\widetilde{\rho}] = \frac{4|t|^2}{t}\big(1+\frac{\rho\tau}{2t}\big)[\mathrm{d}t\wedge \mathrm{d}\bar{t}|\mathrm{d}\tau]. \tag{2.9}$$

But the B insertions (2.2) also change, introducing a compensating Jacobian in (2.4). Finally, changing variables to t and τ also changes the state $\langle\Sigma|$. Equation (1.3) gives

$$\begin{aligned}\langle\Sigma, z-\widetilde{r}+\widetilde{\rho}\theta| =& \langle\Sigma, z-\widetilde{r}|\big(1-2\widetilde{\rho}G_{-\frac{1}{2}}\big) = \langle\Sigma, z-(r+t^2)-t\rho\tau-(\rho-t\tau)\theta| \\ =& \langle\Sigma, z-(r+t^2)|\big(1+t\rho\tau L_{-1}+2(\rho-t\tau)G_{-\frac{1}{2}}\big).\end{aligned} \tag{2.10}$$

In this formula, we see the consequences of the non-split nature of (1.10). The L_{-1} term would not be there for some split transformation. So, though the measure is perfectly continuous, the appearance of the extra term in (2.7) is a consequence of expanding it in coordinates appropriate to the neighborhood of the boundary of moduli space when the two punctures collide.

Recall that our goal is to perform an operator product expansion, which in this context amounts to rewriting the state that arises from the collision of two operators as the insertion of a single operator at the collision point. The result in Green and Seiberg was that the naïve OPE resulted in a state that was not the highest component of a superfield and hence violated supersymmetry. Restoring supersymmetry required adding a contact term. However, as we will shortly see, the state that we find *is* the highest component of a superfield inserted at the collision point, and thus, no contact term is needed. It is precisely the "extra" L_{-1} term that restores supersymmetry. The key point is to appreciate what "taking the OPE" means in the operator formalism: take the two states at P and Q, contract them with the state associated to the 3-punctured sphere, and insert the state that results at the neck where the 3-punctured sphere is joined onto the rest of the surface (Fig. 1). In doing so, it is vital to use the coordinates appropriate to this superconformal geometry. In particular, to see that the state inserted at the neck is the highest component of a superfield, we need to use a contour-deformation, or more pedantically, to make use of the Borel-ambiguity associated to the 3-punctured sphere. From the Neveu-Schwartz algebra, we have

$$L_{-1} = 2\{G_{-\frac{1}{2}}, G_{-\frac{1}{2}}\} = 4G^2_{-\frac{1}{2}}.$$

Thus, (2.7) becomes

$$\begin{aligned}\int \left[\mathrm{d}r \wedge \mathrm{d}\bar{r} \wedge \mathrm{d}t \wedge \mathrm{d}\bar{t}|\mathrm{d}\rho \mathrm{d}\tau\right]\ 4\rho\tau|t|^2 \langle \Sigma, \sigma_{P_0}, \sigma_{Q_0}|\big(G^{(P)}_{-\frac{1}{2}} + G^{(Q)}_{-\frac{1}{2}}\big)G^{(P)}_{-\frac{1}{2}} \\ \times\, b^{(P)}_{-1}\bar{b}^{(P)}_{-1}\delta[\beta^{(P)}_{-\frac{1}{2}}]b^{(Q)}_{-1}\bar{b}^{(Q)}_{-1}\delta[\beta^{(Q)}_{-\frac{1}{2}}]|\psi\rangle^P \otimes |\psi\rangle^Q.\end{aligned} \tag{2.11}$$

The factor of $(G^{(P)}_{-\frac{1}{2}} + G^{(Q)}_{-\frac{1}{2}})$ is equivalent (up to the Borel-ambiguity associated to the 3-punctured sphere) to an insertion $G^{(N)}_{-\frac{1}{2}}$, on the neck of the sphere that is pinching off. Effectively, we can replace the sum of two contours surrounding P and Q with one contour surrounding both and deformed this contour onto the neck (Fig. 2).

Fig. 2:Contour manipulations.

The b and $\delta(\beta)$ insertions have removed the ghost dependence in our SPSs and we are left with a state $|\widetilde{\psi}\rangle_M$ in the matter theory, inserted on the neck, given by

$$|\widetilde{\psi}\rangle_M = G^{(N)}_{-\frac{1}{2}}|\sigma_N\rangle\langle\sigma_P, \sigma_Q|G^{(P)}_{-\frac{1}{2}}|\psi\rangle^P_M \otimes |\psi\rangle^Q_M \quad . \tag{2.12}$$

Here $|\sigma_N\rangle\langle\sigma_P, \sigma_Q|$ is the state associated to the 3-punctured sphere, viewed as an element of $\mathcal{H}\otimes\mathcal{H}^*\otimes\mathcal{H}^*$. Since $|\widetilde{\psi}\rangle_M$ is the highest component of a superfield (*i.e.* it is $G_{-\frac{1}{2}}$ acting on the lowest component of a superfield), it is supersymmetric.

Thus we have maintained world-sheet supersymmetry *without* adding an explicit contact term when the two vertex operators coincide. The correction to the "naïve" result (of taking the OPE of two (0)-picture vertex operators) arises not from an explicit contact term, but rather from a proper account of the superconformal geometry associated to the collision of two punctures. Since the correction is proportional to $L^{(P)}_{-1}$, it looks like a total derivative in the modulus associated to the location of P. Hence its effects can be *simulated* by adding a δ-function contact term at the boundary of moduli space.

3. Concluding Remarks

One slightly puzzling feature of our calculation is that the dependence of (2.6) on the coordinate family used to construct the measure dropped out only for coordinate families with the correct asymptotic behavior. In the bosonic string, one can prove that the measure for SPSs is completely insensitive to the coordinate family chosen. But the argument rests on the fact that Ω is the pullback of a differential form on $\mathcal{P}_{g,n}$ which, for SPSs, has the property that a) it is constant along the fibers of $\mathcal{P}_{g,n} \to \mathcal{M}_{g,n}$ and b) it annihilates the vertical directions. In the super case, however, Ω is in no sense a pullback of a differential form on $\widehat{\mathcal{P}}_{g,n}$. Rather [17], it is a section of the sheaf BER of Berezin forms on $\widehat{\mathcal{M}}_{g,n}$ and has no invariantly-defined precursor "upstairs" on $\widehat{\mathcal{P}}_{g,n}$. So the corresponding argument cannot be made in the super case. Nevertheless, we have seen that merely demanding the correct asymptotic behavior of the coordinate family is enough in this case to remove all dependence on the "shape" of the coordinate family chosen. Perhaps a generalization of the arguments of [7] would allow this to be extended globally over moduli space. If not, then it might still be the case that these unwanted contributions are actually total derivatives in the moduli and, hence, decouple anyway, although that is hard to see explicitly in this particular calculation. Finally, let us note that the possible dependence of the string measure on the coordinate family is quite distinct from the "integration ambiguity" raised in the literature (see [18,19] and references therein) a few years ago. For the integration ambiguity was not really an ambiguity in the measure [20], but rather, an ambiguity in the region of integration. Here the region of integration (defined by some cutoff $|t| > \epsilon$) is the same; it is really the measure that suffers from a potential ambiguity. We believe that there is, in fact, no ambiguity so long as one restricts oneself to coordinate families with the correct asymptotic behavior at infinity (*i.e.* coordinate families which also give the correct dilaton insertions). A more careful investigation of this question would be desirable.

What are the implications of our results for calculating string scattering amplitudes? The simplest scattering amplitude where these issues arise is the 4-point function on the sphere. $\widehat{\mathcal{M}}_{0,4}$ has dimension $1|2$, and the boundary of $\widehat{\mathcal{M}}_{0,4}$ consists of three divisors of dimension $0|2$. Each boundary component corresponds to the sphere pinching in two, with two punctures landing on either side of the pinch. When all of the punctures are far apart, we can choose coordinates for supermoduli space by saying that the four punctures are located at $(0,0)$ $(1,\rho_1)$, $(r,0)$ and (∞,ρ_2). We also demand that the local coordinate at each puncture depend only on the moduli associated to that puncture. Near each of the boundaries of supermoduli space, however, these coordinates degenerate, and must be replaced by those associated to the pinching geometry. The transformations between the coordinates (r, ρ_1, ρ_2) and those appropriate near the boundary must involve a nonsplit transformation for at least one of the components of the boundary. In the case at hand, it is $r \to 0$ which involves the nonsplit transformation but, however we choose to coordinatize the interior of supermoduli space, it is impossible to avoid a nonsplit transformation somewhere [20]. In constructing the corresponding string measure, this nonsplit transformation manifests itself as a correction to the "naïve" measure, which restores the superconformal Ward identities. The calculation of the measure in the interior of moduli space is easily translated into the conventional picture-changing language. The vertex operators at $1, \infty$ are in the (0)-picture, while the vertex operators at $0, r$ are in the (-1)-picture. But the picture-changing is too naïve, and in need of corrections when the two (-1)-picture vertex operators run into each other.

The real point of this is that, in general, we need to cut off the integration because the kinematics may be such that the measure diverges near some boundary of moduli space. If we simply impose a cutoff ($|r| > \epsilon$), in the picture-changing formalism, we obtain a surface term which not superconformally covariant [5]. To it, Green and Seiberg were forced to add a contact term. A better approach is to impose a cutoff on the pinching parameter ($|t| > \epsilon$). The measure constructed in the operator formalism is *automatically* superconformally invariant. With this choice of cutoff, no extra contact term is required. In particular, the contact term was required to remove the contribution of states like the bosonic string tachyon which are not even in the spectrum of the fermionic string theory. In the present formalism, such states are already projected out. The *only* divergences of the string measure in the present formalism are due to *physical* on-shell poles.

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